42 Points Math Olympiad | December 2019

Each participant must work independently and must submit his/her own solutions. The only instruments permitted are pencils, erasers, rulers and compasses. Computers, cell-phones, tablets, calculators, books or any other material is not permitted. The maximum time allowed for the exam is 3 hours. Every problem is worth 7 points, making 28 points the maximum possible score.

Problems for Level 1 (7-8 grades)

1. **Geometry**

   Points D and G are chosen inside the triangle ABC, such that $\angle BDC = 150°$ and $\angle BGC = 100°$. Find the measure of the angle $\angle BAC$ if it is known that $\angle ABG = \angle GBD$ and $\angle ACG = \angle GCD$.

2. **Number Theory**

   Three prime numbers are such that their product is 103 times greater than their sum. Find all such numbers.

3. **Combinatorics**

   There is a pile of 2019 stones on a table. You are allowed to perform the following operation: you choose one of the piles containing more than 1 stone, throw away one stone from that pile and divide it into two smaller (not necessarily equal) piles. Is it possible to reach a situation in which all the piles on the table contain exactly 7 stones?

4. **Algebra**

   Given four positive numbers a, b, c, d and the number x that satisfy the equation

   $$\frac{x-a-b-c}{d} + \frac{x-b-c-d}{a} + \frac{x-c-d-a}{b} + \frac{x-d-a-b}{c} = 4$$

   Find the value of

   $$\frac{a+b+c+d}{x}$$

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Solutions

1. **Geometry**

   Answer: $\angle BAC = 50°$.

   Let $\angle ABG = \angle GBD = x$, $\angle ACG = \angle GCD = y$, $\angle DBC = a$, $\angle DCB = b$. From the triangle BDC: $a + b = 180° - 150° = 30°$. From the triangle BGC: $x + y = 180° - 100° - a - b = 50°$. From the triangle ABC: $\angle BAC = 180° - 2x - 2y - a - b = 180° - 100° - 30° = 50°$.

2. **Number Theory**

   Answer: the only such numbers are 3, 53 and 103.

   Let the numbers be p, q and r. Therefore

   $$pqr = 103(p + q + r)$$

   Notice that 103 is prime. Since the left-hand side is divisible by 103, so should be the right-hand side and therefore one of the numbers is 103. Let this number be p: $p = 103$. The equality now becomes

   $$qr = 103 + q + r$$
   $$qr - q - r + 1 = 104$$
   $$(q - 1)(r - 1) = 104$$

   Since 104 can only be factored as $1 \cdot 104$, $2 \cdot 52$, $4 \cdot 26$ or $8 \cdot 13$, then we have $q = 3$ and $r = 53$ or vice versa.

3. **Combinatorics**

   Answer: it is impossible.

   Assume that it is possible to have exactly n piles with 7 stones each. Notice that after each operation the total number of stones in the piles decreases by 1, while the total number of piles increases by 1. Therefore, the sum of the total number of stones and the total number of piles is invariant. In the beginning the sum is equal to $2019 + 1 = 2020$. In the end the sum is equal $7n + n = 8n$. Since 2020 is not divisible by 8 we obtained a contradiction.

4. **Algebra**

   Answer: 1.

   Rewrite the equation as

   $$\frac{x-a-b-c}{d} - 1 + \frac{x-b-c-d}{a} - 1 + \frac{x-c-d-a}{b} - 1 + \frac{x-d-a-b}{c} - 1 = 0$$
   $$\frac{x-a-b-c-d}{d} + \frac{x-a-b-c-d}{a} + \frac{x-a-b-c-d}{b} + \frac{x-a-b-c-d}{c} = 0$$
   $$(x-a-b-c-d) \cdot \left(\frac{1}{d} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 0$$

   This implies that $x - a - b - c - d = 0$ or equivalently $x = a + b + c + d$. Therefore $\frac{a+b+c+d}{x} = 1$.