42 Points Math Olympiad | December 2019

Each participant must work independently and must submit his/her own solutions. The only instruments permitted are pencils, erasers, rulers and compasses. Computers, cell-phones, tablets, calculators, books or any other material is not permitted. The maximum time allowed for the exam is 3 hours. Every problem is worth 7 points, making 28 points the maximum possible score.

Problems for Level 2 (9-10 grades)

1. **Combinatorics**

   There is a pile of 2019 stones on a table. You are allowed to perform the following operation: you choose one of the piles containing more than 1 stone, throw away one stone from that pile and divide the pile into two smaller (not necessarily equal) piles. Is it possible to reach a situation in which all the piles on the table contain exactly 7 stones?

2. **Algebra**

   Find all positive integer numbers n, such that there exist n positive real numbers x_1, x_2, ... , x_n that satisfy the equalities

$$x_1 + x_2 + ... + x_n = 3$$
$$\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n} = 3$$

3. **Geometry**

   Point M is chosen inside the triangle ABC, such that $\angle BMC = 90^\circ + \frac{1}{2}\angle BAC$. Line AM contains the circumcenter of the triangle BMC. Show that the lines BM and CM contain the circumcenters of the triangles AMC and AMB respectively.

4. **Number Theory**

   Show that there exists an infinite number of positive integers of the form 5^n, such that their decimal representation contains at least 2019 consecutive zeros.

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Solutions

1. **Combinatorics**

   Answer: it is impossible.

   Assume that it is possible to have exactly n piles with 7 stones each. Notice that after each operation the total number of stones in the piles decreases by 1, while the total number of piles increases by 1. Therefore, the sum of the total number of stones and the total number of piles is invariant. In the beginning the sum is equal to $2019 + 1 = 2020$. In the end the sum is equal $7n + n = 8n$. Since 2020 is not divisible by 8 we obtained a contradiction.

2. **Algebra**

   Answer: $n = 2, 3$.

   By multiplying the equations and applying the AM-GM inequality we have

$$9 = (x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \geq n\sqrt[n]{x_1 x_2 ... x_n} \cdot \frac{n}{\sqrt[n]{x_1 x_2 ... x_n}} = n^2$$

   and therefore $n \leq 3$. For $n = 1$ such number x_1 does not exist. For $n = 2$ such numbers exist: $x_1 = \frac{3+\sqrt{5}}{2}$, $x_2 = \frac{3-\sqrt{5}}{2}$. For $n = 3$ such numbers exist: $x_1 = x_2 = x_3 = 1$.

3. **Geometry**

   Let O_A, O_B, O_C be the circumcenters of the triangles BMC, AMC, AMB respectively and let the angles of the triangle ABC be equal α, β, γ. We will show that M is the incenter of the triangle ABC. Let X be any point on the circumcircle of the triangle BMC. We have

$$\angle BO_AC = 2\angle BXC = 2\,(180^\circ - \angle BMC) = 2\left(90^\circ - \frac{1}{2}\angle BAC\right) = 180^\circ - \alpha$$

   which implies that the quadrilateral ABO_AC is cyclic. Since $\angle MO_AC$ is central and $\angle MBC$ is inscribed, we have $\angle MO_AC = 2\angle MBC$. Therefore $\angle ABC = \angle AO_AC = 2\angle MBC$ and MB is the angle bisector of $\angle ABC$. Similarly, MC is the angle bisector of $\angle ACB$ and M is the incenter of the triangle ABC. Therefore

$$\angle AMO_B + \angle AMB = \frac{1}{2}\,(180^\circ - \angle AOM) + 180^\circ - \angle BAM - \angle ABM = 90^\circ - \frac{\beta}{2} + 180^\circ - \frac{\alpha}{2} - \frac{\gamma}{2} = 180^\circ$$

   and the points O_B, M, B lie on the same line. Similarly, the points O_C, M, C lie on the same line.

4. **Number Theory**

   Let us show that for any $k \in \mathbb{N}$, there exist infinitely many $m \in \mathbb{N}$, such that $5^m \equiv 1 \pmod{2^k}$. Consider the sequence $5^0, 5^1, 5^2, \dots, 5^{2^k}$. By Pigeonhole principle there are two numbers 5^p and 5^q $(p > q)$ that have the same remainders mod 2^k. Therefore $5^p - 5^q = 5^q(5^{p-q} - 1)$ is divisible by 2^k and then $5^{p-q} \equiv 1 \pmod{2^k}$. Since $\left(5^{p-q}\right)^i \equiv 1^i \equiv 1 \pmod{2^k}$, then for $m = (p-q)i$, $i \in \mathbb{N}$, we have $5^m \equiv 1 \pmod{2^k}$. Therefore $5^{m+k} \equiv 5^k \pmod{10^k}$, which means that the last k digits of the number 5^{m+k} are exactly the same as the number 5^k with possibly some zeros in front of it. For $2^k > 10^{2019}$ we have that $5^k = 10^k/2^k < 10^k/10^{2019} = 10^{k-2019}$ has no more than $k - 2019$ digits. Therefore from the last k digits of the number 5^{m+k} only the last $k - 2019$ digits are non-zero. Therefore the rest 2019 digits should be zero digits.