42 Points Math Olympiad | December 2019

Each participant must work independently and must submit his/her own solutions. The only instruments permitted are pencils, erasers, rulers and compasses. Computers, cell-phones, tablets, calculators, books or any other material is not permitted. The maximum time allowed for the exam is 3 hours. Every problem is worth 7 points, making 28 points the maximum possible score.

Problems for Level 3 (11-12 grades)

1. **Combinatorics**

   Given an infinite square paper and a set N of $4n$ randomly chosen squares. Is it always possible to choose at least n squares from the set N, such that no two of the chosen squares share a common point?

2. **Algebra**

   The plane is tiled with congruent equilateral triangles of side 1. Let d_1 be the distance between some vertices A_1 and B_1 and d_2 be the distance between some vertices A_2 and B_2. Is it possible to find the vertices A_3 and B_3, such that the distance between A_3 and B_3 is equal to the product $d_1 \cdot d_2$?

3. **Number Theory**

   Show that there exists an infinite number of positive integers of the form 5^n, such that their decimal representation contains at least 2019 consecutive zeros.

4. **Geometry**

   Let ω be the circumcircle of the triangle ABC. Point H is chosen on the angle bisector of the angle $\angle ABC$ and point D is chosen on ω, such that $DB \parallel AC$. DH and BH intersect ω at points P and Q, respectively. The line passing through H and parallel to AC intersects AB and BC at the points R and S, respectively. Prove that the quadrilateral $PQRS$ is cyclic.

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Solutions

1. **Combinatorics**

   Answer: yes, it is possible.

   We will paint all the squares of the plane in 4 colors: black, white, blue and yellow. We take any row and paint it black, white, black, white, etc. Then we take the row below it and paint it blue, yellow, blue, yellow, etc. We continue this process until all the plane is painted. By Pigeonhole Principle at least n squares from the set N are of the same color. Since the squares of the same color do not share a point, we can choose them to be the needed n squares.

2. **Algebra**

   Answer: yes, it is possible.

   Consider two unit vectors e_1 and e_2 that form $120°$ and are collinear with the sides of the triangles. Let $\overrightarrow{A_1B_1} = ae_1 + be_2$ and $\overrightarrow{A_2B_2} = ce_1 + de_2$ for some integers a, b, c, d. Then $d_1^2 = \overrightarrow{A_1B_1} \cdot \overrightarrow{A_1B_1} = a^2 - ab + b^2$ and $d_2^2 = \overrightarrow{A_2B_2} \cdot \overrightarrow{A_2B_2} = c^2 - cd + d^2$. However, since $d_1^2 \cdot d_2^2 = \left(a^2 - ab + b^2\right)\left(c^2 - cd + d^2\right) = x^2 - xy + y^2$ for $x = ac - bd$ and $y = ad + bc - db$, we have that the vector $\overrightarrow{A_3B_3} = xe_1 + ye_2$ satisfies the conditions of the problem.

3. **Number Theory**

   By Euler's Theorem, for any $k \in \mathbb{N}$: $5^{\phi(2^k)} \equiv 1 \pmod{2^k}$. Therefore for all m of the form $\phi(2^k)i$, $i \in \mathbb{N}$, we have $5^m \equiv 1 \pmod{2^k}$. Therefore $5^{m+k} \equiv 5^k \pmod{10^k}$, which means that the last k digits of the number 5^{m+k} are exactly the same as the number 5^k with possibly some zeros in front of it. For $2^k > 10^{2019}$ we have that $5^k = 10^k/2^k < 10^k/10^{2019} = 10^{k-2019}$ has no more than $k - 2019$ digits. Therefore from the last k digits of the number 5^{m+k} only the last $k - 2019$ digits are non-zero. Therefore the rest 2019 digits should be zero digits.

4. **Geometry**

   Consider the inversion around the point B: points A^*, C^*, Q^*, D^* and P^* are collinear, point H^* is the midpoint of the arc S^*R^*, the circumcircles of the triangles BC^*A^* and BR^*S^* are tangent to the line D^*B. Therefore there exists a homothecy that takes the circumcircle of the triangle BC^*A^* into the circumcircle of the triangle BR^*S^*. Then $P^*Q^* \parallel S^*R^*$. Let X be the intersection of the circumcircle of the triangle BR^*S^* and the line H^*P^*. We have $\angle H^*P^*Q^* = \angle H^*BD^* = \angle H^*R^*B = \angle H^*BX$, which implies that $P^*Q^* \parallel BX$ and $PQRS$ is an isosceles trapezoid and therefore is cyclic.