42 Points Math Battle | April 2019

Each team must work independently and must present only their own solutions. The only instruments permitted are pencils, erasers, rulers and compasses. Computers, cell-phones, tablets, calculators, books or any other material is not permitted. The maximum time allowed for the exam is 3 hours.

Problems

Captains Contest

The difference of squares of two consecutive integer numbers is 2019. What are the numbers?

Problem 1

A grasshopper jumps along a line. His first jump takes him 1 cm, his second 2 cm, his third 3 cm, and so on. Each jump can take him to the right or to the left. Show that after 2017 jumps the grasshopper cannot return to the point at which he started.

Problem 2

Given that a, b and $\sqrt{a}+\sqrt{b}$ are rational numbers. Show that $\sqrt{a}$ and $\sqrt{b}$ are also rational numbers.

Problem 3

Given some natural number n. Show that the numbers $n(n-1)$ and $(n+1)^2$ have different sums of their digits.

Problem 4

Given 10 integer numbers. Show that it is possible to choose some of these numbers and place the signs "+" and "−" between them in such way that the result is divisible by 1001.

Problem 5

Show that $2^n + 1$ is not divisible by 7 for any natural n.

Problem 6

Given a circle ω and two fixed points A and B on the circle, such that the measure of the arc AB is $120°$. Point C slides along on the arc AB, H and O are the orthocenter and the circumcenter of the triangle ABC. Show that the perpendicular bisectors to the segments HO pass through a common point.

Problem 7

Given that

$$a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

Show that if $abc = 1$, then one of the numbers is equal to 1.

Problem 8

Lines AP, BP and CP intersect the sides of the triangle ABC at the points A_1, B_1 and C_1 respectively. Prove that the lines passing through the midpoints of the sides BC, AC and AB and the midpoints of the AA_1, BB_1 and CC_1 are concurrent.

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Solutions

Captains Contest

Answer: 1009 and 1010.
Indeed, let the number be n and $n+1$. Therefore $(n+1)^2 - n^2 = 2n+1$, which implies that $n = 1009$ and $n+1 = 1010$.

Problem 1

Let the grasshopper start at the origin. Notice that the parity of the grosshoppers' position is odd after $4k+1$ jumps. Since $2019 \equiv 1 \pmod{4}$, then the grasshopper cannot return to the point at which he started.

Problem 2

Consider the equality $\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right) = a-b$. Since a, b and $\sqrt{a}+\sqrt{b}$ are rational then $\sqrt{a}-\sqrt{b}$ is also rational. Therefore $\left(\sqrt{a}-\sqrt{b}\right)+\left(\sqrt{a}+\sqrt{b}\right) = 2\sqrt{a}$ is also rational and $\left(\sqrt{a}-\sqrt{b}\right)-\left(\sqrt{a}+\sqrt{b}\right) = -2\sqrt{b}$ is also rational. This implies that $\sqrt{a}$ and $\sqrt{b}$ are also rational numbers.

Problem 3

Let the given numbers have equal sums of their digits. It is known that the sum of the digit of a number should be congruent to the number modulus 3. However $(n+1)^2 = n^2+2n+1 \equiv n^2-n+1 = n(n-1)-1 \not\equiv n(n-1) \pmod{3}$.
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Problem 4

Consider all possible subsets of the original set: there are 2^{10} possible such subsets. Let us consider the remainders modulo 1001 of the sums of the numbers in each subset: there are $2^{10} > 1001$ sums. By Pigeonhole Principle there exist two subsets that have the same remainders modulo 1001. Now we can eliminate their common elements and place the signs "+" in front of the numbers of one of the subsets and "−" in front of the numbers of the other subset.

Problem 5

Note that $8^n \equiv 1^n \equiv 1 \pmod 7$. For $n = 3k$: $2^n = 2^{3k} = 8^k \equiv 1 \pmod 7$. For $n = 3k+1$: $2^n = 2^{3k+1} = 8^k \cdot 2 \equiv 1 \cdot 2 \equiv 2 \pmod 7$. For $n = 3k+2$: $2^n = 2^{3k+2} = 8^k \cdot 4 \equiv 1 \cdot 4 \equiv 4 \pmod 7$. We conclude that 8^n is not divisible by 7 for any natural n.

Problem 6

Note that $\angle ACB = 60°$ and $\angle AOB = 120°$. Since $\angle AHB = 120°$ and therefore the quadrilateral $AOHB$ lie on the same circle, let's say ω. Since HO is the chord on the circle ω, then its perpendicular bisector passes through the center of the circle, which is a fixed point.

Problem 7

Note that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab+bc+ac}{abc} = ab + bc + ac$ and therefore $a + b + c = ab + bc + ac$. Now consider the expression: $(1-a)(1-b)(1-c) = 1 - (a+b+c) + (ab+bc+ac) - abc = 0$, which implies that one of the numbers should equal to 1.

Problem 8

Let us apply the Ceva's Theorem to the lines AP, BP and CP: $\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1$. Let the midpoints of the sides AB, BC and AC C_2, A_2 and B_2 respectively and the midpoints of the segments AA_1, AA_1, AA_1 be A_3, B_3 and C_3 respectively. Note that A_3, B_3 and C_3 lie on the sides of the triangle $A_2B_2C_2$. The conclusion of the problem follows from the application of the Ceva's Theorem to the lines A_2A_3, B_2B_3 and C_2C_3.