42 Points Math Olympiad | April 2020

Each participant must work independently and must submit only his/her own solutions. The only instruments permitted are pencils, erasers, rulers and compasses. Any books or math related material is not permitted during the solution process. The maximum time allowed for the exam is 4 hours. Additional 30 minutes are allowed for the digitalization and submission purposes. Every problem is worth 7 points, making 28 points the maximum possible score.

Problems for Level 1 (7-8 grades)

1. **Geometry**

   Given a triangle ABC. Point D is chosen on the side BC, such that $AD = BD$. Find the measure of the angle $\angle ADC$ if it is known that $\angle DAC = \angle ABC - \angle ACB$.

2. **Combinatorics**

   3 squares of size 2×2 are cut off from the board 5×5. Is it always possible to cut off one more 2×2 square?

3. **Algebra**

   Find all real numbers a and b, such that

   $$\frac{a}{b} = \frac{b}{b-a}$$

4. **Number Theory**

   Show that the number

   $$N = 1^1 + 2^2 + ... + 2019^{2019} + 2020^{2020}$$

   is divisible by 3.

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Solutions

1. **Geometry**

   Answer: $\angle ADB = 120°$.

   Let $\angle ABC = \beta$ and $\angle ACB = \gamma$. Note that we are given that $\angle DAC = \beta - \gamma$. The triangle ADB is isosceles and therefore $\angle BAD = \angle ABD = \beta$. Form here $\angle ADB = 180° - 2\beta$ and $\angle ADC = 2\beta$. From the triangle ADC we have $(2\beta) + (\beta - \gamma) + (\gamma) = 180°$ or $\beta = 60°$ and $\angle ADC = 120°$.

2. **Combinatorics**

   Answer: yes, it is possible.

   Paint 4 corner squares of size 2×2 in black. Notice that when we cut a 2×2 square it can only affect at most one black square. Therefore by Pigeonhole Principle there exists a black square that will not be affected and we will be able to cut it off as a 2×2 square.

3. **Algebra**

   Answer: such numbers do not exist.

   The equation is equivalent to $a(b-a) = b^2$. This implies that $a^2 + b^2 = ab$. If $ab < 0$, then the left-hand side is positive and the right-hand side is negative. Contradiction. If $ab > 0$, then by AM-GM $a^2 + b^2 \geq 2\sqrt{a^2b^2} = 2ab > ab$ and the left-hand side is greater than the right-hand side. Contradiction. Therefore $ab = 0$. If $a = 0$, then the initial equation becomes $0 = \frac{b}{b}$, which has no solutions. If $b = 0$, then the expression on the left-hand side is undefined.

4. **Number Theory**

   Note that $(6k)^{6k} \equiv 0 \pmod 3$, $(6k+1)^{6k+1} \equiv (1)^{6k+1} \equiv 1 \pmod 3$, $(6k+2)^{6k+2} \equiv (-1)^{6k+2} \equiv 1 \pmod 3$, $(6k+3)^{6k+3} \equiv 0 \pmod 3$, $(6k+4)^{6k+4} \equiv (1)^{6k+4} \equiv 1 \pmod 3$, $(6k+5)^{6k+5} \equiv (-1)^{6k+5} \equiv -1 \pmod 3$. Therefore every block of 6 consecutive terms is congruent to $0 + 1 + 1 + 0 + 1 + (-1) \equiv 2 \pmod 3$ and thus the sum of 18 consecutive terms is divisible by 3. Since $2020 = 2016 + 4 = 18 \cdot 112 + 4$, then $N \equiv 2017^{2017} + 2018^{2018} + 2019^{2019} + 2020^{2020} \equiv 1 + 1 + 0 + 1 \equiv 0 \pmod 3$.