42 Points Math Olympiad | April 2020

Each participant must work independently and must submit only his/her own solutions. The only instruments permitted are pencils, erasers, rulers and compasses. Any books or math related material is not permitted during the solution process. The maximum time allowed for the exam is 4 hours. Additional 30 minutes are allowed for the digitalization and submission purposes. Every problem is worth 7 points, making 28 points the maximum possible score.

Problems for Level 2 (9-10 grades)

1. **Number Theory**

   Find all distinct prime numbers p and q, such that the number

   $$(p+q)^2 + (p-q)^2$$

   is a multiple of pq.

2. **Geometry**

   Let ABC be a triangle with $AB > AC$ and let k be its circumcircle. The line tangent to the circle k at the point A intersects the line BC at the point P. Let m be a line passing through the point P and intersecting the sides AB and AC at the points D and E respectively, such that $AD = AE$. Show that the line m is the angle bisector of the angle $\angle APB$.

3. **Algebra**

   Given that

   $$\frac{a}{b+c} = \frac{b}{c+a} = \frac{c}{a+b}$$

   Find all possible values of the expression

   $$S = \frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b}$$

4. **Combinatorics**

   9999 squares of size 2×2 are cut off from the board 299×299. Is it always possible to cut off one more 2×2 square?

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Solutions

1. **Number Theory**

   Answer: there are no such prime numbers.

   Note that $(p+q)^2+(p-q)^2 = 2p^2+2q^2$. Since $p \mid 2p^2+2q^2$, then $p \mid 2q^2$. Since $p \neq q$, then $p = 2$. Similarly $q \mid 2p^2+2q^2$, then $q \mid 2p^2$. Since $p \neq q$, then $q = 2$. However since $p \neq q$, then there are no such prime numbers.

2. **Geometry**

   Let $\angle ABC = \alpha$ and $\angle ADE = \beta$. Since $AD = AE$, then the triangle ADE is isosceles and $\angle AED = \angle ADE = \beta$. Since AP is tangent to k, then $\angle CAP = \angle CBA = \alpha$. The angle $\angle AEP = 180^\circ - \beta$ and thus $\angle EPA = \beta - \alpha$. The angle $\angle BDP = 180^\circ - \beta$ and thus $\angle BPD = \beta - \alpha$. Therefore $\angle APD = \angle BPD$ and the line m is the angle bisector of the angle $\angle APB$.

3. **Algebra**

   Answer: $S = -3$ or $S = 6$.

   From the first equality we have $a(a+c) = b(b+c)$ or equivalently $(a-b)(a+b+c) = 0$. From the second equality we have $b(a+b) = c(a+c)$ or equivalently $(b-c)(a+b+c) = 0$. If $a+b+c = 0$, then $a+b = -c$, $b+c = -a$, $a+c = -b$ and $S = -1-1-1 = -3$. If $a+b+c \neq 0$, then $a-b = 0$ and $b-c = 0$, which implies that $a = b = c$ and $S = 2+2+2 = 6$.

4. **Combinatorics**

   Answer: yes, it is possible.

   Paint 10000 squares of size 2×2 in black color in such way so the black squares are 1 square apart form each other and they do not share a common point. Notice that when we cut a 2×2 square it can only affect at most one black square. Therefore at most 9999 of these squares will be affected. By Pigeonhole Principle there exists a black square that will not be affected and we will be able to cut it off.