42 Points Math Olympiad | April 2020

Each participant must work independently and must submit only his/her own solutions. The only instruments permitted are pencils, erasers, rulers and compasses. Any books or math related material is not permitted during the solution process. The maximum time allowed for the exam is 4 hours. Additional 30 minutes are allowed for the digitalization and submission purposes. Every problem is worth 7 points, making 28 points the maximum possible score.

Problems for Level 3 (11-12 grades)

1. **Number Theory**

   Let p_1, p_2, ... , p_n be distinct prime numbers and P be their product. Let the number S be defined as

   $$S = \sum_{i=1}^{n} \left(\frac{P}{p_i}\right)^{p_i - 1}$$

   Show that $S - 1$ is a multiple of P.

2. **Geometry**

   Line ω passes through the orthocenter H of an acute triangle ABC. Let ω_A, ω_B and ω_C be the reflections of the line ω across the sides BC, AC and AB respectively. Prove that ω_A, ω_B and ω_C are concurrent.

3. **Combinatorics**

   A positive integer number n is written on the board. It is allowed to take any written number m, erase it and then write two positive integer numbers x and y, such that $xy = 2m^2$. Prove that if at some moment there are 100 numbers written on the board, then one of them is not greater than $10n$.

4. **Algebra**

   Given the positive numbers a, b and c, such that $abc = 8$. Prove that

   $$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}$$

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Solutions

1. **Number Theory**

   Notice that $\frac{P}{p_i}$ is an integer and $\gcd\left(\frac{P}{p_i}, p_i\right) = 1$ for all $i = 1, 2, ..., n$. By Fermat's Theorem: $\left(\frac{P}{p_i}\right)^{p_i - 1} \equiv 1 \pmod{p_i}$. Notice that $\frac{P}{p_i}$ is divisible by p_j for all $i \neq j$. Therefore $S \equiv 1 \pmod{p_i}$ and $p_i | (S-1)$ for all $i = 1, 2, ..., n$. Since p_i are all distinct, then $S - 1$ is divisible by their product and $P | (S - 1)$.

2. **Geometry**

   Without loss of generality let us assume that the line ω intersects the sides AB and BC at the interior points M and N respectively. Let A_1, B_1, C_1 be the reflections of the orthocenter H across the lines BC, AC, AB respectively. Let H_A, H_B, H_C be the bases of the altitudes from A, B, C to the sides BC, AC, AB respectively. Since A_1, B_1, C_1 belong to the circumcircle of the triangle ABC, then MC_1 is the line ω_C and NA_1 is the line ω_A. Let X_A and X_C be the points of intersection of the lines MC_1 and NA_1 with the circumcircle of the triangle ABC. We have $\angle AA_1X_A = \angle HA_1N = \angle NHA_1$ and $\angle CC_1X_C = \angle HC_1M = \angle MHC_1$. Since M, H, N are collinear and BH_CHH_A is cyclic, then $\angle NHA_1 + \angle MHC_1 = \angle ABC$. Therefore $\angle AA_1X_A + \angle CC_1X_C = \angle ABC$ and the points X_A and X_C coincide (let us say at the same point X). This implies that the lines ω_A and ω_C intersect on the circumcircle of the triangle ABC. Since $\angle MHB_1 = \angle XB_1H$, then the line XB_1 is symmetric to ω with respect to AC and ω_A, ω_B and ω_C are concurrent at X.

3. **Combinatorics**

   Let x_1, x_2, ... , x_m be the numbers on the board after $m-1$ operations. Consider the sums of squares of the reciprocals of these numbers: $S_m = \sum_{i=1}^{m}\left(\frac{1}{x_i^2}\right)$. From $xy = 2n^2$ we have $\frac{1}{2n^2} = \frac{1}{xy} \leq \frac{1}{2}\left(\frac{1}{x^2} + \frac{1}{y^2}\right)$ by AM-GM inequality. From here $\frac{1}{n^2} \leq \frac{1}{x^2} + \frac{1}{y^2}$ and therefore $S_1 \leq S_2 \leq ... \leq S_m$ for all $m \in \mathbb{N}$. Let z be the smallest number on the board after 99 operations, i.e. $z \leq x_i$ for all $i = 1, 2, ..., 100$. Therefore $\frac{1}{n^2} = S_1 \leq S_2 \leq ... \leq S_{100} \leq 100 \cdot \frac{1}{z^2}$ and thus $z \leq 10n$.

4. **Algebra**

   Let the left-hand side of the inequality be S. Notice that $\frac{1}{\sqrt{1+x^3}} \geq \frac{2}{2+x^2}$ and therefore $S \geq \frac{4a^2}{(2+a^2)(2+b^2)} + \frac{4b^2}{(2+b^2)(2+c^2)} + \frac{4c^2}{(2+c^2)(2+a^2)} = \frac{4a^2(2+c^2)+4b^2(2+a^2)+4c^2(2+b^2)}{(2+a^2)(2+b^2)(2+c^2)} = \frac{2A}{36+A}$, where $A = 2a^2 + 2b^2 + 2c^2 + a^2b^2 + b^2c^2 + a^2c^2$. Taking into account $abc = 8$ and using the AM-GM inequality we have $a^2 + b^2 + c^2 \geq 12$ and $a^2b^2 + b^2c^2 + a^2c^2 \geq 48$. Therefore we have $A \geq 72$, which is equivalent to $\frac{2A}{36+A} \geq \frac{4}{3}$.