42 Points Math Battle | April 2018

Each team must work independently and must present only their own solutions. The only instruments permitted are pencils, erasers, rulers and compasses. Computers, cell-phones, tablets, calculators, books or any other material is not permitted. The maximum time allowed for the exam is 3 hours.

Problems

Captains Contest

Let $S(n)$ be the sum of the digits of the number n. What is the smallest natural number n, such that $S(n)$ and $S(n+1)$ are both divisible by 8?

Problem 1

There are two piles of stones: 2017 and 2018 stones in each. At each turn, a player may take as many stones as he chooses, but only from one of the piles. The loser is the player who cannot make a move. Who will win the game?

Problem 2

201 points are placed inside a 1×1 square, such that no three of them lie on the same line. Is it true that some three of them form a triangle with area less than 0.01?

Problem 3

The 8×8 board is divided into horizontal rectangles 2×1 and vertical rectangles 1×2. Is it possible that there are exactly 15 horizontal and 17 vertical rectangles?

Problem 4

There are 2015 red, 2017 green and 2019 blue chameleons on the Rainbow Island. When two chameleons of different colors meet they both change their color to the third one (for instance, if blue and green meet they will both become red). Is it possible that after some time all the chameleons on the island are of the same color?

Problem 5

In an acute triangle ABC the measure of the angle B is equal 60°. The altitudes CE and AD intersect at the point H. Show that the circumcenter O of the triangle ABC lies on the common angle bisector of the angles $\angle AHE$ and $\angle CHD$.

Problem 6

Find all integer numbers x, y and z, such that

$$x^2 + y^2 = 3z^2$$

Problem 7

Let a, b, c, d be positive real numbers such that $a + b + c + d = 1$. Show that

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2}$$

Problem 8

Point P is chosen on the side AD of the trapezoid $ABCD$ ($AB \parallel CD$), such that $AB = AP$ and $DC = DP$. Q is the midpoint of the side BC and R is the intersection of the circumcircle of the triangle AQB with the side AD. Show that the points B, C, P and R lie on the same circle.

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Solutions

Captains Contest

Answer: $n = 79$.
Indeed, $s(79) = 16$, $s(80) = 8$ and it is not hard to see that $n = 79$ is the smallest such number.

Problem 1

Answer: first player wins.
The first player should take one stone out of the pile with 2018 stones and then mirror the second player's turns by taking the same amount of stones from a different pile.

Problem 2

Answer: it is true.
Let us divide a square into 100 squares 0.1×0.1. Then by Pigeonhole Principle there exists a square with at least three points inside, which form a triangle with area less than 0.01.

Problem 3

Answer: it is impossible.
Let us paint the first column of the board in black, the second in white, etc. The horizontal rectangles cover exactly 15 black squares. Therefore the other 17 black squares should be covered by the vertical rectangles, which is impossible since each vertical rectangle covers an even number of black squares.

Problem 4

Answer: it is impossible.

After two chameleons meet the triple representing the number of chameleons of each color (x, y, z) is changed to $(x+1, y+1, z-2)$. Therefore the triple of remainders of (x, y, z) is invariant modulo 3. The initial triple of remainders is $(0, 1, 2)$ and the final is not.

Problem 5

Since $\angle AHC = \angle DHE = 120°$ and $\angle AOC = 2\angle ABC = 120°$, then $AOHC$ is cyclic. Since $\angle ACH = \angle ACO = 30°$, then O lies on the angle bisector of the angle $\angle AHE$.

Problem 6

Answer: $x = y = z = 0$.

The triple $(0, 0, 0)$ clearly satisfies the equation. Let us assume that $(x, y, z) \neq (0, 0, 0)$. If $d = \gcd(x, y, z)$ and $x = dx_1$, $y = dy_1$, $z = dz_1$, then $x_1^2 + y_1^2 = 3z_1^2$ and $\gcd(x_1, y_1, z_1) = 1$. Since $x_1^2 + y_1^2$ is congruent to 0, 1 or 2 modulo 4 and $3z_1^2$ is congruent to 0 or 3 modulo 4, then both sides of the equations should be congruent to 0 modulo 4, which holds only when x_1, y_1, z_1 are all even and contradicts $\gcd(x_1, y_1, z_1) = 1$.

Problem 7

Let $u = \left(\sqrt{a+b}, \sqrt{b+c}, \sqrt{c+d}, \sqrt{d+a}\right)$ and $v = \left(\frac{a}{\sqrt{a+b}}, \frac{b}{\sqrt{b+c}}, \frac{c}{\sqrt{c+d}}, \frac{d}{\sqrt{d+a}}\right)$. Let us apply the Cauchy-Schwarz inequality to the vectors u and v:

$$((a+b) + (b+c) + (c+d) + (d+a)) \left(\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a}\right) \geq (a+b+c+d)^2 = 1$$

which is equivalent to the statement of the problem.

Problem 8

Notice that $\angle BPC = 180° - \angle APB - \angle DPC = 180° - \frac{1}{2}(180° - \angle ADC) - \frac{1}{2}(180° - \angle DAB) = \frac{1}{2}(\angle ADC + \angle DAB)$. Therefore $\angle BPC = 90°$ and Q is the circumcenter of the triangle BPC $QC = QP = QB$. Triangles ABQ and APQ are congruent by side-side-side congruency and thus $\angle ABQ = \angle APQ$. Since $ABQR$ is cyclic $\angle ABQ = DRQ$. This implies that the triangle PQR is isosceles and therefore $PQ = RQ$ and the point R is concyclic with the points B, P, C.